Linear Algebra I

26/01/2016, Monday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 $(5+4+2+2+2=15 \text{ pts})$ **Linear equations**

Let A and b be given as

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}.$$

Determine

(a) the row echelon form of A.

(b) the reduced row echelon form of A.

(c) the rank of A.

(d) the set of solutions of the homogeneous system $Ax = 0$.

(e) the set of solutions of the system $Ax = b$.

REQUIRED KNOWLEDGE: **Gauss-elimination, row operations, set of solutions, and rank.**

SOLUTION:

1a: By applying row operations, we obtain:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 \end{bmatrix} \xrightarrow[\textbf{4th} = \textbf{4th} - \textbf{1st}]{\substack{\textbf{2nd} = \textbf{2nd} - \textbf{1st} \\ \textbf{3rd} = \textbf{3rd} - \textbf{1st}}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 7 \end{bmatrix} \xrightarrow{\textbf{2nd} = \frac{1}{2}\textbf{2nd}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 7 \end{bmatrix} \xrightarrow{\textbf{4th} = \textbf{4th} - \textbf{2nd}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{\substack{\textbf{3nd} = \textbf{4th} \\ \textbf{4th} = \textbf{3nd}}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\textbf{4th} = \frac{1}{2}\textbf{4th}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1b: To obtain the row reduced echelon form, we continue with the row operations

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\textbf{3rd} = \textbf{3rd} - 4 \times \textbf{4th} \\ \textbf{2nd} = \textbf{2nd} - 3 \times \textbf{4th} \\ \textbf{1st} = \textbf{1st} - \textbf{4th}}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\textbf{2nd} = \textbf{2nd} - 2 \times \textbf{3rd} \\ \textbf{1st} = \textbf{1st} - \textbf{3rd}}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{\textbf{1st} = \textbf{1st} - \textbf{2nd}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

1c: As the rank equals to the number of leading zeros in the row (reduced) echelon form, we see that $\text{rank}(A) = 4$.

1d: Since the row echelon form is a nonsingular matrix, the only solution to $Ax = 0$ is the trivial solution $x = 0$.

1e: In order to find the set of solutions, we apply the same row operation in the same order to the vector b:

$$\begin{bmatrix}0\\0\\2\\3\end{bmatrix}\xrightarrow[]{\substack{\textbf{2nd} = \textbf{2nd} - \textbf{1st}\\ \textbf{3rd} = \textbf{3rd} - \textbf{1st}\\ \textbf{4th} = \textbf{4th} - \textbf{1st}}}\begin{bmatrix}0\\0\\2\\3\end{bmatrix}\xrightarrow{\textbf{2nd} = \frac{1}{2}\textbf{2nd}}\begin{bmatrix}0\\0\\2\\3\end{bmatrix}\xrightarrow{\textbf{4th} = \textbf{4th} - \textbf{2nd}}\begin{bmatrix}0\\0\\2\\3\end{bmatrix}\xrightarrow[]{\substack{\textbf{3nd} = \textbf{4th}\\ \textbf{4th} = \textbf{3nd}}}\begin{bmatrix}0\\0\\3\\2\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\3\\2\end{bmatrix}\xrightarrow{\textbf{4th} = \frac{1}{2}\textbf{4th}}\begin{bmatrix}0\\0\\3\\1\end{bmatrix}\xrightarrow[]{\substack{\textbf{3rd} = \textbf{3rd} - 4\times\textbf{4th}\\ \textbf{2nd} = \textbf{2nd} - 3\times\textbf{4th}\\ \textbf{1st} = \textbf{1st} - \textbf{4th}}}\begin{bmatrix}-1\\-3\\-1\\1\end{bmatrix}\xrightarrow[]{\substack{\textbf{2nd} = \textbf{2nd} - 2\times\textbf{3rd}\\ \textbf{1st} = \textbf{1st} - \textbf{3rd}}}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\xrightarrow{\textbf{1st} = \textbf{1st} - \textbf{2nd}}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}.$$

Therefore, $Ax = b$ has the unique solution

$$x = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}.$$

Let A and B be square matrices.

(a) Show that $(A - B)(A + B) = A^2 - B^2$ if and only if $AB = BA$.

(b) Suppose that both A and B are nonsingular. Show that $A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1}$.

(c) Suppose that $A + B$ is nonsingular. Show that $A(A + B)^{-1}B = B(A + B)^{-1}A$.

REQUIRED KNOWLEDGE: **Matrix multiplication and inverse.**

SOLUTION:

2a: Direct calculations yiled:

$$
\begin{aligned}
(A - B)(A + B) = A^2 - B^2 \quad &\iff \quad A^2 + AB - BA - B^2 = A^2 - B^2 \\
&\iff \quad AB - BA = 0 \\
&\iff \quad AB = BA.
\end{aligned}
$$

2b: Note that

$$
\begin{aligned}
A^{-1}(A + B)B^{-1} &= A^{-1}AB^{-1} + A^{-1}BB^{-1} \\
&= B^{-1} + A^{-1} = A^{-1} + B^{-1}.
\end{aligned}
$$

2c: Note that

$$A(A + B)^{-1}B + A(A + B)^{-1}A = A(A + B)^{-1}(B + A) = A(A + B)^{-1}(A + B) = A$$

and

$$B(A + B)^{-1}A + A(A + B)^{-1}A = (B + A)(A + B)^{-1}A = (A + B)(A + B)^{-1}A = A.$$

Consequently, we obtain

$$A(A + B)^{-1}B = B(A + B)^{-1}A.$$

Consider the matrix

$$M = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

(a) Determine the determinant of M.

(b) For which values of a, b, and c, is the matrix M nonsingular?

(c) Determine all values of a, b, and c such that $\text{rank}(M) = 2$.

(d) Determine the characteristic polynomial of M.

(e) Find the eigenvalues of M.

(f) Without computing the eigenvectors, determine whether M is diagonalizable.

REQUIRED KNOWLEDGE: **Determinant, nonsingularity, rank, eigenvalues, and diagonalization.**

SOLUTION:

3a: Note that

$$\det(M) = \det(\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}) = a\det(\begin{bmatrix} a & b \\ -c & 0 \end{bmatrix}) - b\det(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) = abc - abc = 0.$$

3b: A square matrix is nonsingular if and only if its determinant is nonzero. So, the matrix M is singular regardless of the values of a, b, and c.

3c: From 3b, it is clear that $\text{rank}(M) \leqslant 2$. Note that the vectors

$$\begin{bmatrix} 0 \\ -a \\ -b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a \\ 0 \\ -c \end{bmatrix}$$

are linearly independent if and only if $a \neq 0$. Similarly, the vectors

$$\begin{bmatrix} a \\ 0 \\ -c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$$

are linearly independent if and only if $c \neq 0$. Finally, the vectors

$$\begin{bmatrix} b \\ c \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -a \\ -b \end{bmatrix}$$

are linearly independent if and only if $b \neq 0$. Then, we have

$$a \neq 0 \quad \text{or} \quad b \neq 0 \quad \text{or} \quad c \neq 0 \implies \text{rank}(M) = 2.$$

Now, we claim that

$$\text{rank}(M) = 2 \implies a \neq 0 \quad \text{or} \quad b \neq 0 \quad \text{or} \quad c \neq 0.$$

To see this, suppose, on the contrary, that $a = b = c = 0$. Then, we have $M = 0$ and hence $\text{rank}(M) = 0$.

Consequently, we obtain

$$\text{rank}(M) = 2 \iff a \neq 0 \quad \text{or} \quad b \neq 0 \quad \text{or} \quad c \neq 0.$$

3d: Characteristic polynomial of M can be found as

$$\det(M - \lambda I) = \det \begin{bmatrix} -\lambda & a & b \\ -a & -\lambda & c \\ -b & -c & -\lambda \end{bmatrix} = -\lambda^3 + abc - abc - b^2\lambda - c^2\lambda - a^2\lambda = -\lambda^3 - (a^2 + b^2 + c^2)\lambda.$$

3e: Eigenvalues are the roots of the characteristic polynomial. Note that $\lambda^3 + (a^2 + b^2 + c^2)\lambda = 0$ gives $\lambda_1 = 0$ and $\lambda_{2,3} = \pm i\sqrt{a^2 + b^2 + c^2}$.

3f: We know that a matrix is diagonalizable if it has distinct eigenvalues. Since M has distinct eigenvalues, it is diagonalizable.

Consider the vector space P_4. Let $S \subseteq P_4$ be defined as

$$S := \{p(x) \in P_4 \mid p(x) = x^3 p(\frac{1}{x})\}.$$

(a) Show that S is a subspace of P_4.

(b) Find a basis for S.

(c) Find the dimension of S.

Let $T : P_4 \to P_4$ be defined as

$$T\big(p(x)\big) := p(x) + x^2 p'(\frac{1}{x})$$

where $p'(x)$ denotes the derivative of $p(x)$.

(d) Show that T is a linear transformation.

(e) Determine $\ker(T)$.

(f) Find the matrix representation of T with respect to the ordered basis $\{1, x, x^2, x^3\}$.

REQUIRED KNOWLEDGE: **Subspaces, basis, dimension, linear transformations, and their matrix representations.**

SOLUTION:

4a: Note that

i. S is nonempty: The zero polynomial belongs to S.

ii. S is closed under scalar multiplication: Let α be a scalar and $p(x) \in S$. Also, let $q(x) = \alpha p(x)$. Note that $q(x) = \alpha p(x) = \alpha x^3 p(\frac{1}{x}) = x^3 \alpha p(\frac{1}{x}) = x^3 q(\frac{1}{x})$. Thus, $q(x) \in S$.

iii. S is closed under vector addition: Let $p(x)$ and $q(x)$ belong to S. Also, let $r(x) = p(x)+q(x)$. Note that $r(x) = p(x) + q(x) = x^3 p(\frac{1}{x}) + x^3 q(\frac{1}{x}) = x^3\big(p(\frac{1}{x}) + q(\frac{1}{x})\big) = x^3 r(\frac{1}{x})$. Hence, $r(x) \in S$.

Consequently, S is a subspace.

4b: Let $p(x) = a + bx + cx^2 + dx^3$ belong to S. Since $p(x) = x^3 p(\frac{1}{x})$, we have

$$a + bx + cx^2 + dx^3 = x^3(a + b\frac{1}{x} + c\frac{1}{x^2} + d\frac{1}{x^3}) = d + cx + bx^2 + ax^3.$$

This means that $a = d$ and $b = c$. Thus, $p(x)$ belongs to S if and only if $p(x) = a + bx + bx^2 + ax^3$. Note that $a + bx + bx^2 + ax^3 = a(1 + x^3) + b(x + x^2)$. From this observation, it follows that $1 + x^3$ and $x + x^2$ span S. Since they are also linearly independent, we can conclude that the vectors $1 + x^3$ and $x + x^2$ form a basis for S.

4c: The dimension of a subspace equals to the cardinality (the number of elements) of the basis. Therefore, the dimension of S is 2.

4d: Let α be a scalar and $p(x)$, $q(x) \in P_4$. Note that

$$T\big(\alpha p(x)\big) = \alpha p(x) + x^2 (\alpha p)'(\frac{1}{x}) = \alpha p(x) + x^2 \alpha p'(\frac{1}{x}) = \alpha\big(p(x) + x^2 p'(\frac{1}{x})\big) = \alpha T\big(p(x)\big)$$

and

$$
\begin{aligned}
T\big(p(x)+q(x)\big) &= p(x)+q(x)+x^2\big(p+q\big)'(\frac{1}{x}) \\
&= p(x)+q(x)+x^2\big(p'(\frac{1}{x})+q'(\frac{1}{x})\big) \\
&= p(x)+x^2p'(\frac{1}{x})+q(x)+x^2q'(\frac{1}{x}) \\
&= T\big(p(x)\big)+T\big(q(x)\big).
\end{aligned}
$$

Therefore, T is a linear transformation.

4e: Let $p(x)=a+bx+cx^2+dx^3 \in \ker T$. Then, we have

$$0=T\big(p(x)\big)=p(x)+x^2p'(\frac{1}{x})=a+bx+cx^2+dx^3+x^2(b+2c\frac{1}{x}+3d\frac{1}{x^2})=(a+3d)+(b+2c)x+(b+c)x^2+dx^3.$$

Therefore, $p(x)=a+bx+cx^2+dx^3 \in \ker T$ if and only if $a+3d=b+2c=b+c=d=0$. By solving this equations, we obtain $a=b=c=d=0$. Consequently, $\ker T=\{0\}$.

4f: To determine the matrix representation, we apply the transformation to the basis vectors:

$$
\begin{aligned}
T(1) &= 1+x^2\cdot 0 &&= 1\cdot 1+0\cdot x+0\cdot x^2+0\cdot x^3 \\
T(x) &= x+x^2\cdot 1 &&= 0\cdot 1+1\cdot x+1\cdot x^2+0\cdot x^3 \\
T(x^2) &= x^2+x^2(2\frac{1}{x})=x^2+2x &&= 0\cdot 1+2\cdot x+1\cdot x^2+0\cdot x^3 \\
T(x^3) &= x^3+x^2(3\frac{1}{x^2})=x^3+3 &&= 3\cdot 1+0\cdot x+0\cdot x^2+1\cdot x^3
\end{aligned}
$$

This leads to the matrix representation:

$$
\begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

Find the parabola $y = a + bx + cx^2$ that gives the best least squares approximation to the points:

x	-1	0	1	2
y	1	-1	0	2

REQUIRED KNOWLEDGE: **Least squares, and normal equations.**

SOLUTION:

We can formulate the following least squares problem in order to find the best parabola of the given type:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

The normal equations of the least square problem $Ax = b$ are given by

$$A^T Ax = A^T b.$$

Note that

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}.$$

Thus, the normal equations are given by:

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}.$$

By applying Gauss elimination, we obtain

$$\left[\begin{array}{ccc|c} 4 & 2 & 6 & 2 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 9 \end{array}\right] \xrightarrow[\textbf{3rd} = \textbf{3rd} - 6 \times \textbf{1st}]{\textbf{1st} = \frac{1}{4} \times \textbf{1st},\ \textbf{2nd} = \textbf{2nd} - 2 \times \textbf{1st}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 5 & 5 & 2 \\ 0 & 5 & 9 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 5 & 5 & 2 \\ 0 & 5 & 9 & 6 \end{array}\right] \xrightarrow{\textbf{3rd} = \textbf{3rd} - \textbf{2nd}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 5 & 5 & 2 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 5 & 5 & 2 \\ 0 & 0 & 4 & 4 \end{array}\right] \xrightarrow{\textbf{3rd} = \frac{1}{5} \times \textbf{3rd},\ \textbf{4th} = \frac{1}{4} \times \textbf{4th}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 1 & \frac{2}{5} \\ 0 & 0 & 1 & 1 \end{array}\right].$$

This leads to the solution

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{7}{10} \\ -\frac{3}{5} \\ 1 \end{bmatrix}.$$

Therefore, the best parabola is given by

$$y = -\frac{7}{10} - \frac{3}{5}x + x^2.$$

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

(a) Show that 1 is an eigenvalue of A.

(b) Determine all eigenvalues of A.

(c) Find a matrix T such that $T^{-1}AT$ is diagonal.

REQUIRED KNOWLEDGE: **Eigenvalues, eigenvectors, and diagonalization.**

SOLUTION:

6a: The matrix A has an eigenvalue λ if and only if $\det(A - \lambda I) = 0$. Note that

$$\det(A - I) = \det(\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}) = 0$$

as the first column consists entirely of zeros. Therefore, 1 is an eigenvalue of A.

6b: Note that

$$\det(A - \lambda I) = \det(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{bmatrix}) = (1-\lambda)\det(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}) = (1-\lambda)(\lambda^2+1).$$

This results in the eigenvalues $\lambda_1 = 1$, $\lambda_{2,3} = \pm i$.

6c: In order to find a diagonalizer, we need to compute an eigenvector for each eigenvalue:

i. $\lambda_1 = 1$: Note that

$$0 = (A - I)x_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} x_1$$

results in

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

ii. $\lambda_2 = i$: Note that

$$0 = (A - iI)x_2 = \begin{bmatrix} 1-i & 1 & 1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} x_2$$

results in

$$x_2 = \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}.$$

iii. $\lambda_3 = -i$: Note that

$$0 = (A - iI)x_3 = \begin{bmatrix} 1+i & 1 & 1 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix} x_3$$

results in

$$x_3 = \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}.$$

Therefore, one can take

$$T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & -i & i \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix}.$$

To verify that T is a diagonalizer, note that

$$AT = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -i & i \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & i & -i \\ 0 & -1 & -1 \end{bmatrix}$$

and

$$T \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & -i & i \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & i & -i \\ 0 & -1 & -1 \end{bmatrix}.$$